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Algebra! The cry to teach and learn algebra is being heard across the nation. The National Math Advisory Panel, the National Council of Teachers of Mathematics (NCTM), and even politicians are declaring a mandate for every U.S. student to learn algebra in order to better prepare for the world of work in the years ahead. This year, in fact, the Adult Numeracy Network (ANN) chose algebra as its central topic for discussion at its Annual Meeting (a COABE pre-session on April 18 in Louisville, KY). ANN recognizes that adult education students, many of whom aspire to further training or college work, may be able to pass the GED without understanding the mathematics of relationships and equations; but are likely to find 'next steps' challenging, if not impossible, without the mathematics of algebra.

In order to be effective algebra teachers, many adult education math teachers need to better understand the complex and ever-expanding field of mathematics we call "algebra."

In this issue, we attempt to share some basic information about algebra, including historical background (p. 6), information about variables (p. 5) and equations (p.4) and on pp. 1-4 we share some of Mark Driscoll's thoughts about algebraic thinking and the questions essential to developing it.

We hope in future issues to continue reviewing research documents to explore algebra, an exploration that humans have pursued for millennia. Please, join us on the journey.
Tricia Donovan, Editor

In This Issue

Algebraic Thinking and the Questions That Nurture It

All excerpts are from: Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers grades 6-10*, Heinemann, Portsmouth, NH.

"Because algebra comprises so many mathematical features, the term algebraic thinking defies simple definition," admits Driscoll in his opening chapter, "Developing Algebraic Habits of Mind: A Framework for Classroom Questions Aimed at Understanding Student Thinking." However, he continues describing the perspective he developed through his work with grades 6-10 teachers, a perspective that emphasizes "habits of thinking that can begin developing in the prealgebra years and, if nurtured, can serve the learning of formal algebra as well." Though there are many potential habits of algebraic thinking, Driscoll lists three habits "that seem to be critical:" (1) doing and undoing, (2) building rules to represent functions, and (3) abstracting from computation.

The three habits Driscoll considers crucial to algebraic thinking are described thusly:

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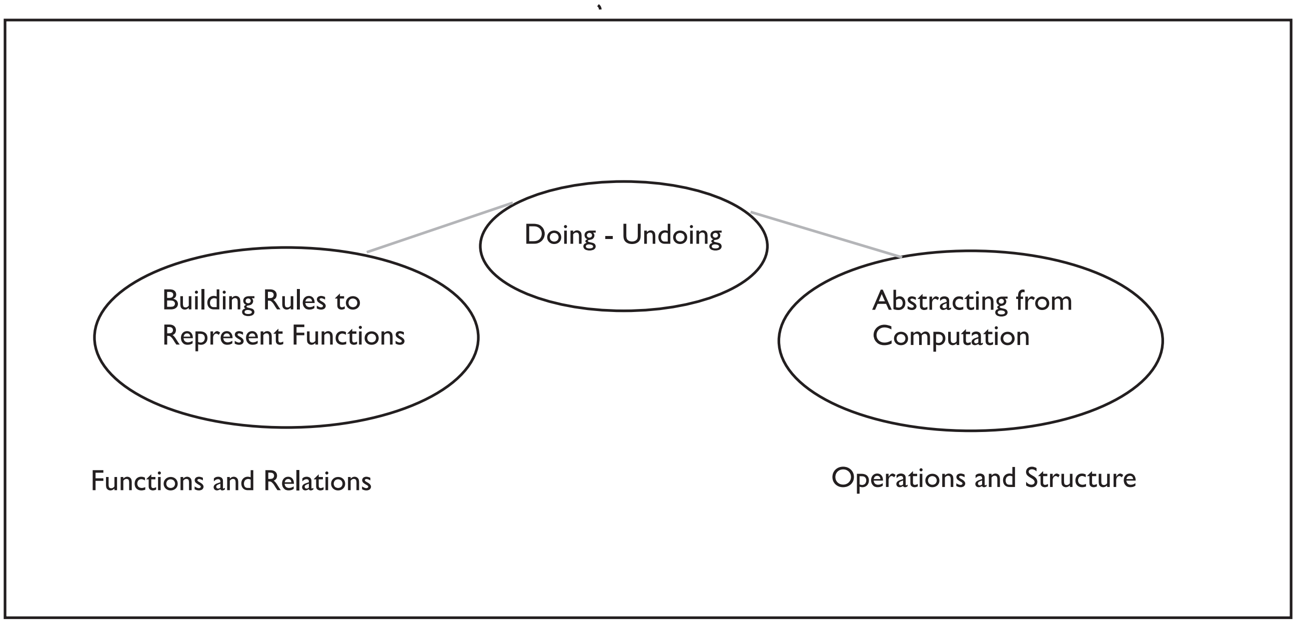
Doing-Undoing. Effective algebraic thinking sometimes involves reversibility (i.e., being able to undo mathematical processes as well as do them). In effect, it is the capacity to not only use a process to get to a goal, but also to understand the process well enough to work backward from the answer to the starting point. So, for example, in a traditional algebraic setting, algebraic thinkers can not only solve an equation such as 9x2 – 18 = 0, but also answer the question, "What is an equation with solutions 4/3 and -4/3?"

Building Rules to Represent Functions. Critical to algebraic thinking is the capacity to recognize patterns and organize data to represent situations in which input is related to output by well-defined functional rules. For example, here is a functional rule that is computation-based: "Take an input number, multiply it by 4 and subtract 3." This habit of mind is a natural complement to Doing-Undoing, in that the capacity to understand how a functional rule works in reverse generally makes it a more accessible and useful process.

Abstracting from Computation. This is the capacity to think about computations independently of particular numbers that are used. One of the most evident characteristics of algebra has always been its abstractness. But, just what is being abstracted? To answer this, a good case can be made that thinking algebraically involves being able to think about computations freed from the particular numbers they are tied to in arithmetic—that is, abstracting system regularities from computation. For example, students invoke this habit of mind when they realize that they can regroup numbers into pairs that equal 101 to make the following computation simpler: "Compute 1 + 2 + 3 + … + 100" There is a suggestion of Doing-Undoing here, as well, in the recognition that 101 can be decomposed into 100 + 1; 99 + 2; 98 + 3; and so on. (Driscoll, 1999, pp. 1-2)

To develop these habits of mind, Driscoll and fellow researchers advise teachers to provide "consistent modeling of algebraic thinking," and to give "well-timed pointers to students that help them shift or expand their thinking or that help them pay attention to what is important," and make it a "habit to ask a variety of questions aimed at helping students organize their thinking and respond to algebraic prompts." (See page 13)



Adapted from Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers grades 6-10.* Portsmouth, NH: Heinemann.

Driscoll associates each habit of mind with specific classroom questions that help foster algebraic thinking:

Habit of Mind	Prompting Question
Doing and Undoing	• How is this number in the sequence related to the one that came before? • What if I start at the end? • Which process reverses the one I'm using? • Can I decompose this number or expression into helpful components?
Building Rules to Represent Functions	• Is there a rule or relationship here? • How does the rule work, and how is it helpful? • Why does the rule work the way it does? • How are things changing? • Is there information here that lets me predict what's going to happen? • Does my rule work for all cases? • What steps am I doing over and over? • Can I write down a mechanical rule that ill do this job once and for all? • How can I describe the steps without using specific inputs? • When I do the same thing with different num bers, what still holds true? What changes? • Now that I have an equation, how do the numbers (parameters) in the equation relate to the problem context?
Abstracting From Computation	• How is this calculating situation like/unlike that one? • How can I predict what's going to happen without doing all the calculation?

Algebraic Thinking...

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Driscoll also categorizes teachers' questions as: managing, clarifying, orienting, prompting mathematical reflection and eliciting algebraic thinking. An example or two of each follows:

- Managing – Who's in charge of writing it down?
- Clarifying - Do you know what perimeter is? How did you get '2'?
- Orienting – What's the problem asking you to find? Have you thought about trying a table?
- Prompting Mathematical Reflection -- How do you explain that? Why did the two of you reach different conclusions?
- Eliciting Algebraic Thinking – What could it (the value in the equation) represent? How could you use the formula?

Thinking algebraically, as Driscoll demonstrates, sets the stage for expressing, calculating and solving algebraically. And while Driscoll's work centers on students in grades 6-10 levels, recent work from other researchers makes it clear that algebraic reasoning can begin at even earlier levels. This means students at all NRS levels can and should be exposed to the kinds of problems and questions that foster algebraic thinking.

Editor's Recommendation: Mark Driscoll's book, *Fostering Algebraic Thinking: A Guide for Teachers Grades 6-10,* comes highly recommended as a basic resource for adult educators. Driscoll outlines key elements of algebraic thinking in a readable fashion. He peppers his book with examples of student and teacher work/thinking and includes interesting problems and activities from which practitioners may draw ideas for their own classrooms. You will find yourself returning to this text again and again as you deepen your own knowledge of algebra and your students' thinking.

Psychological Complexity of Equations

There are many types of equations, and the role of the variable may change in each. Researcher Daniel Chazan outlined five types of equations in his article, "On Appreciating the Cognitive Complexity of School Algebra: Research on Algebra Learning and Directions of Curricular Change." He sees the ambiguity of concepts, like "equation," leading to student confusion and misunderstanding.

Below, we outline the five types and share what Chazan says about each.

"Zalman Usiskin (1988) presents five equations (in the sense of having an equal sign in them) involving literal symbols. Each has a different feel. …

1. $A = LW$
2. $40 = 5x$
3. $\text{Sin } x = \cos x \cdot \tan x$
4. $1 = n \cdot 1/n$
5. $y = kx$

As Usiskin observes, 'We usually call (1) a formula, (2) an equation or open sentence to solve, (3) an identity, (4) a property, and (5) an equation of a function direct variation not to be solved.' (p.9)

Although readers might agree or disagree with his classification of these strings, that is part of his (Usiskin's) point. To the enculturated, each of these symbol strings reads differently, and perhaps differently enculturated readers read them differently as well. Furthermore, Usiskin (1988) argues that each of these equations has a different feel because in each case the idea of variable is put to a different use:

"In (1), *A, L*, and *W* stand for the quantities area, length, and width and have the feel of knowns. In (2), we tend to think of *x* as an unknown. In (3), *x* is an argument of a function. Equation (4), unlike the others, generalizes an arithmetic pattern, and *n* identifies an instance of the pattern. In (5), *x* is again an argument of a function, *y* the value, and *k* a constant (or parameter, depending on how it is used). Only with (5) is there the feel of "variability," from which the term variable arose. "(p.9) Chazan (2003), p. 125

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Chazan, D. (2003). "On appreciating the Cognitive Complexity of School Algebra: Research on Algebra Learning and Directions of Curricular Change," Chapter 9 *A Research Companion to Principles and Standards for School Mathematics*, Jeremy Kilpatrick W. Gary Martin, and Deborah Schifter, Eds., National Council of Teachers of Mathematics, Reston, VA, pp. 123-135.

7 Different Uses of Literal Symbols

What is a variable? Randolph Philipp points out in his article "The Many Uses of Algebraic Variables" that 'variables' or literal symbols, play many roles in mathematics. Sometimes, as in the case of π, they do not vary at all, in fact, they are 'constants,' always equal to the same ratio (in this case 22/7).

He believes that students would be greatly helped in their understanding of algebra if teachers explicitly acknowledged the different forms. This helps students better understand equations, especially equations where different literal symbols play different roles (see Note below).

Note: Philipp's reference to 'varying quantities' touches on the concept of 'functions.' He uses the term 'parameters', to refer to the cost per mile charged for a rented car, for instance. So in equation number five below, y equals the total number of dollars charged for the rental, x stands for the number of miles traveled, m stands for the cost per mile – a parameter in the problem -- and b stands for the number of dollars charged for the standard of time rented – a constant.

1.	Labels	f, y in $3f = 1y$ (3 feet in 1 yard)
2.	Constants	π, e, c
3.	Unknowns	x in $5x - 9 = 91$
4.	Generalized numbers	a, b in $a + b = b + a$
5.	Varying quantities	x, y in $y = 9x - 2$
6.	Parameters	m, b in $y = mx + b$
7.	Abstract symbols	e, x in $e * x = x$

from Table 1 in Philipp, 1992)

Philipp, R.A. (1992). "The Many Uses of Algebraic Variables," The Mathematics Teacher, NCTM, Reston, VA, Vol. 85, Iss. 7, Oct. 1992, p. 557.

What is Algebra?

825 AD – al-jabr,
"the science of restoration and balancing" in order to solve equations.

The term first appears in Mohammed ibn Musa al-Khwarizmi's book *The Comprehensive Book of Calculation by Balance and Opposition.*

Algebra is a branch of mathematics concerning the study of:

- structure,
- relation, and
- quantity

Algebra Time Line

- Circa 1800 BC – Babylonian tablets mention quadratic elliptic equations.
- Circa 600 BC – Indian mathematician Apastamba solves the general linear equation.
- Circa 300 BC – Greek *Euclid's Elements* addresses algebraic concepts through geometry.
- Circa 100 BC – Chinese mathematicians solve systems of simultaneous linear equations.
- Circa 200 AD – Alexandrian Greek Diophantus writes *Arithmetica*, featuring solutions of algebraic equations.
- Circa 7th Century AD – Indian mathematicians use a form of algebraic notation using letters of the alphabet and other signs and give a general algebraic formula for the quadratic equation.
- 820 AD – Persian mathematician Muhammad ibn Musa al-Kwarizmi writes "*The Compendius Book on Calculation by Completion and Balancing* – sets up algebra as an independent mathematics discipline – uses the term 'al-jabr'.
- 990 AD -- Persian mathematician Al-Karaji replaces geometrical operations of algebra with modern arithmetical operations, which are at the core of algebra today.
- 1202 -- Algebra is introduced to Europe largely through the work of Leonardo Fibonacci of Pisa in his work *Liber Abaci.*
- 1412-1482 – Arab mathematician takes the first steps toward the introduction of algebraic symbolism.
- 1591 -- Francois Viete develops improved symbolic notation for various powers of an unknown and uses vowels for unknowns and consonants for constants in *In artem analyticam isagoge*.
- 1682 -- Gottfried Wilhelm Leibniz develops his notion of symbolic manipulation with formal rules which he calls characteristica generalis. Leibniz also introduced the terms *variable, constant, parameter, function, and coordinate*.
- Retrieved and adapted from "http://en.wikipedia.org/wiki/Timeline of algebra" and Earliest Known Uses of Some of the Words of Mathematics retrieved from http://jeff560.tripod.com/mathword.html

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Algebra, the institution, "alienates even the nominally successful students from genuine mathematical experience, prevents real reform and acts as an engine of inequity for an egregiously high number of students, especially those who are the less advantaged of our society."

—James J. Kaput, formerly of the University of Massachusetts, Dartmouth

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When you hear the word algebra, what comes to mind? A one- or two-year course focusing on manipulating symbols? Well, algebra is much more than that! One of the biggest challenges facing us as mathematics teachers is to show all students…that algebra is a tool for understanding and describing relationships in widely varied settings. Making connections from descriptions in words, graphs, or tables to symbolic representations brings insight to students. Seeing these links strengthens students' ability to move back and forth between the concrete and the abstract and boosts their confidence in using symbols that they understand to be firmly based on mathematical properties and connected to the world. … Our challenge is to give all students the necessary preparation and opportunities to make learning algebra a successful experience.

—NCTM President Henry S. Kepner, Jr., President's Message NCTM News Bulletin, December 2008