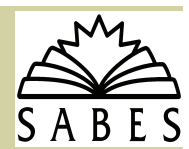# SABES Math Bulletin

Note from the Editor

Many months have passed since the last issue of the *SABES Math Bulletin* was published. During that time, we decided to change the focus of the *Math Bulletin* from research reporting only to a hybrid of research reporting and classroom-related materials. With that decision in mind, we offer you in this issue an assortment of problems for the classroom (see pages 2, 8, and 11) as well as some research-based insights into best mathematics classroom practices.

In this issue, we discuss the value of good classroom questioning (p. 4), the development of algebraic reasoning by fostering the "habit of not knowing" during early stages of arithmetic learning (p. 1) and a brief comparison of the GED and the ACCUPLACER exams (p. 10). We hope you enjoy the look and the content of the new *SABES Math Bulletin*.

Please, if you have any questions or comments, or want to submit some examples of problem solutions that surface in your classroom, feel free to submit them to the *SABES Math Bulletin* at: pdonovan@worlded.org.

Thank you, and enjoy!
Tricia Donovan, Ed.D., Editor

The Habit of Not Knowing: The Key to Reasoning Development

"Isn't mathematics about certainty?

… But how is it we get to certainty and can we do so without expecting not to know?"

The above questions surfaced for Susan Jo Russell of Technical Education Research Centers (TERC) in her work with students of low-level mathematics (primarily working with whole numbers and fractions) and were presented during her session on *Mathematical Reasoning; The Habit of Not Knowing* at the recent Boston Regional National Council of Teachers of Mathematics (NCTM) Conference. Russell polled the audience to demonstrate that most participants viewed "knowing the answer" as the hallmark of the smart math student. Russell begged to differ.

Every Day ABE Teachers Research

Every time a teacher shares a problem and solutions in class, she performs informal research. Her research unearths adults' approaches to mathematics. From the responses she witnesses a range of cognitive applications. Such was the case with the "Who Ran How Far?" problem shared with math teachers through the SABES *Algebra for All* series.

First we share the problem then four solutions. In closing, we ask that you, the reader, consider what these solutions reveal. That's the start of research in the classroom, a task in which you can participate every day.

Naturally, we suggest you first try solving the problem yourself or with a friend.

Who Ran How Far?

Rich mathematical problems often offer a variety of entry points that can be examined to show different mathematics at work. Who Ran How Far? is that kind of problem.

(From the October 2007 issue of *Mathematics Teaching in the   Middle School* 'Solve It' section.)

Aaron, Beth, Candace, Darvel and Emil ran around a track as part of a fundraiser for their school. Aaron ran first, and then when he was tired, Beth ran. When Beth got tired, Candace ran, and so on. By the end of the fundraiser, they had run a total of 30 miles. Aaron and Beth combined ran 40 percent of the total distance. Beth and Candace combined ran 34 percent of the total distance. Candace and Darvel combined ran 40 percent of the total distance. And Darvel and Emil ran 41 percent of the total distance. How far did each person run?

On p. 3 we share four solutions to the problem. You will notice that people assigned letters to correspond to the names in the problem in order to work more efficiently, and so we use these letters to correspond with names throughout the solutions: Aaron (*a*); Beth (*b*), Candace (*c*), Darbel (*d*) and Emil (*e*). Think about the mathematics that each solver uses. Which solution seems algebraic? Which solution seems most simple?

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Solution 1

Aaron and Beth ran 40% or 12 miles

Beth and Candace ran 34% or 10.2 miles

Candace and Darvel ran 40% or 12 miles

Darvel and Emil ran 41% or 12.3 miles

Aaron = a

Beth = $12 - a$

Candace = $10.2 -$ Beth

$\quad\quad 10.2 - (12 - a) = {}^-1.8 + a$

Darvel = $12-$ Candace

$\quad\quad 12 - ({}^-1.8 + a) = 13.8 - a$

Emil = $12.3 -$ Darvel

$\quad\quad 12.3 - (13.8 - a) = {}^-1.5 + a$

The total of all five runners is 30 miles, so

$a + 12 - a + {}^-1.8 + a + 13.8 - a + {}^-1.5 + a = 30$

$a + 22.5 = 30$

Aaron = 7.5 miles

Beth ran $12 - 7.5 = 4.5$ miles

Candace ran $10.2 - 4.5 = 5.7$ miles

Darvel ran $12 - 5.7 = 6.3$ miles

Emil ran $12.3 - 6.3 = 6$ miles

$7.5 + 4.5 + 5.7 + 6.3 + 6 = 30$ miles

Solution 2

Thought: If I can get it down to one letter, I can solve it.

$a + b + c + d + e = 30$

$a + b = 12$	$\longrightarrow$ $b = 12 - a$
$b + c = 10.2$	$\longrightarrow$ $12 - a + c = 10.2$
$c + d = 12$	$\longrightarrow$ $a - 1.8 + d = 12$
$d + e = 12.3$	$\longrightarrow$ $13.8 - a + e = 12.3$

$a + (12 - a) + (a - 1.8) + (13.8 - a) + (a - 1.5) = 30$

$\cancel{a} + 12 - \cancel{a} + \cancel{a} - 1.8 + 13.8 - \cancel{a} + a - 1.5) = 30$

$a = 30 - (12 - 1.8 + 13.8 - 1.5)$

$a = 7.5$

$b = 12 - 7.5 = 4.5$

$c = 10.2 - 4.5 = 5.7$

$d = 12 - 5.7 = 6.3$

$e = 12.3 - 6.3 = 6$

Solution 3

$a + b + c + d + e = 30$

$a + b + c + d = 24$

$30 - 24 = 6$ therefore $e = 6$

If $e = 6$, then

$d = 12.3 - 6 = 6.3$

$c = 12 - 6.3 = 5.7$

$b = 10.2 - 5.7 = 4.5$

$a = 12 - 4.5 = 7.5$

OR

$a + b + c + d + e = 100\%$

$a + b + c + d = 80\%$

$e = 20\%$ and 20% of 30 miles = 6 miles, so $e = 6$ miles

(see solution above for analogous final steps)

Solution 4

$a + b = 40\%$	$c + d = 40\%$
$b + c = 34\%$	$d + e = 41\%$

$a + b = 40\%$	$b + c = 34\%$	$c + d = 40\%$	$d + e = 41\%$
$b = 40 - a$	$40 - a + c = 34$	$a - 6 + d = 40$	$46 - a + e = 41$
	$c = 34 - 40 + a$	$d = 40 + 6 - a$	$e = 41 - 46 + a$
	$c = a - 6$	$d = 46 - a$	$e = a - 5$

$a + b + c + d + e = 100$

$\cancel{a} + 40 - \cancel{a} + a - 6 + 46 - \cancel{a} + \cancel{a} - 5 = 100$

$a + 75 = 100$

$a = 25\%$

$a = 25\%$	7.5 miles
$b = 15\%$	4.5 miles
$c = 19\%$	5.7 miles
$d = 21\%$	6.3 miles
$e = 20\%$	6 miles
$100\% =$	30 miles

Questioning: Motivating Discourse in the Mathematics Classroom

If we want students to talk math, we need to ask a variety of questions that push thinking and require evidence-based responses. During her NCTM Boston Regional Conference presentation, educational consultant Paula Sennet shared several interesting thoughts and practices related to questioning in the mathematics classroom, including increased wait time for responses. Questioning, it turns out, lies at the heart of classroom discourse.
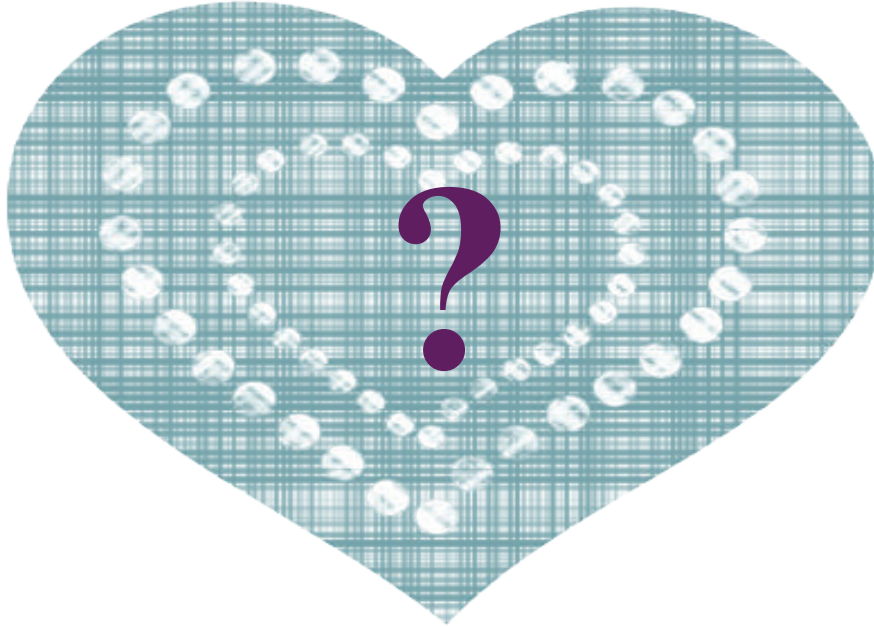
Paraphrasing research from J.S. Sutton and A. Kruger's 2002 paper on *ED Thoughts: What We Know About Mathematics Teaching and Learnin*g (Aurora, CO: Mid-continent Research for Education and Learning), Sennet stated that:

Classroom Questioning

Effective mathematics teachers who are highly rated by their students and whose students perform well on both content and problem-solving assessments ask many questions of all types during their lessons. Compared to less effective teachers, they pose more questions with higher cognitive demand, and ask more follow-up questions. Their students ask more questions, as well.

She lamented that many classrooms operate within a framework where questioning is undervalued.

"Studies of typical mathematics classrooms confirm that most questions make minimal demands or are leading or rhetorical questions and those answered immediately by the teacher."

In addition, she writes, "Answers are often immediately judged right or wrong by the teacher, and discussion moves to the next question."

Seven Question Purposes and Samples

Question Purpose	Sample Questions
1. Building student confidence	*How did you reach that conclusion?* *How can you model that?* *Why is that true?* *Does that make sense?*
2. Learning mathematical reasoning	*Is that true for all cases? Explain.* *How would you prove that?* *Can you think of a counterexample?*
3. Checking progress	*Can you explain what you have done so far? What else is there to do?* *Why did you decide to use this method?* *Can you think of a more efficient strategy?* *Why did you decide to organize your results like that?* *How do you think this would work with other numbers?* *What do you notice when….?*
4. Encouraging conjecturing	*What would happen if….?* *Do you see a pattern? Explain it.* *What are some possibilities here?* *How could you predict the next one? What about the last one?* *What decision do you think he/she should make?*
5. Promoting problem solving	*What do you need to find out?* *What strategies are you going to use?* *What information do you have?* *Will a calculator help? How?* *What tools will you need?* *What do you think the answer or result will be?*
6. Helping students collectively make sense of mathematics	*What do you think about what … said?* *Do you agree or disagree? Why or why not?* *Does anyone have the same answer but a different way to explain it?* *Can you convince the rest of us that your answer makes sense?*
7. Generating reflection	*How did you get your answer?* *What if you could only use…?* *Does your answer seem reasonable? Why or why not?* *Can you describe your method to us all? Can you explain why it works?* *What have you learned or found out today?* *What are the key points or big ideas in this lesson?* *What if you had started with…rather than …?*

Questioning: Motivating Discourse in the Mathematics Classroom

Sennet explained that good discussions take time as she urged participant teachers to:

- Share with students reasons for asking questions.

- Teach for success (don't use questions to embarrass or punish).

- Be nonjudgmental about a response or comment.

- Try not to repeat students' answers.

- Remember: As soon as you tell a student that the answer is correct, thinking stops.

Furthermore, Sennet decried classrooms where the teacher "dominates the interaction using a rapid fire pace and lower cognitive level questions," and where teachers typically "wait less than one second after posing a question before either repeating the question, commenting on a student answer, redirecting the question to a new student, answering the questions themselves or starting a new questioning sequence. "The message students receive," she explained "is that the teacher's way of knowing is the only way of knowing."

Question Purposes

Sennet suggested seven key purposes for asking questions: 1) to help build confidence in solutions; 2) to help learn mathematical reasoning; 3) to check progress; 4) to encourage conjecturing; 5) to promote problem solving; 6) to help the class collaboratively make sense of mathematics and 7) to generate reflection on the mathematics and work.

She adapted materials from the PBS Teacher Source to share the following information which we adapted into a new format and share on p. 5.

Benefits of Good Questioning

Drawing upon research conducted by Dr. Mary Budd Rowe in the 1960's, Sennet shared five 'ripple effects' that occur when teachers allow greater time (at least three seconds) for students to respond, using pauses purposefully:

1. Positive changes in the affective climate.

2. Positive changes in the quality of classroom interactions.

3. An increased level of cognitive functioning (Bloom's Taxonomy).

4. An increased level of academic achievement.

5. A decreased number of behavior problems.

Restructuring Math Class

What does all this mean for ABE classrooms? Sennet outlines several important classroom implications that follow from this research. She suggests that teachers:

- Plan questions while preparing lessons. Write out lesson launching questions and clarifying questions to use during exploration times.

- Choose different questions for varied purposes – clarifying questions, redirection questions, summarizing questions, extension questions, and reflection questions.

- Listen carefully to student answers.

- Assume that every answer given by a student is meaningful and 'correct' to that student. The answers give insight into the student's mind by illuminating misconceptions and misunderstandings.

- Ask for a paraphrase of what has been said. This improves attentiveness and assesses comprehension.

- Allocate sufficient time for thoughtful explanations and dialogue.

- Begin classes with rich problems or questions that engage learners and lead to new understanding of important content.

- Increase wait time for answers.

- Tape your class once in awhile to examine your and your students' questioning.

- Invite in an observant teacher partner to note your 'wait time' practices.

The Habit of Not Knowing: The Key to Reasoning Development
Continued from page 1

Russell asserted that it is essential for mathematics students to develop a "toleration for not knowing and a belief that we can find our way through any problem" if they are to develop the kind of reasoning that will serve them well as they continue to pursue mathematics, whether for school or for work. Mathematicians, she explained, struggle with what they do not know and use what they do know to try to make sense of the problem before them. In particular, Russell and fellow researchers Deborah Shifter and Virginia Bastable are interested in the development of algebraic reasoning, which they and others agree provides the foundation for ongoing school and workplace endeavors.

> *Mathematicians struggle with what they do not know and use what they do know to try to make sense of the problem...*

Linking to Algebra

Russell, Shifter and Bastable see the "underpinnings" of algebraic reasoning and problem solving embedded in development of the early stages of computational fluency. At the earliest stages of mathematical practice, they believe, learners either engage as mathematicians seeking an understanding of what they do not know, or they grow to think math is magic that some people get and some do not. Through work on computational fluency, Russell, Shifter, and Bastable have begun to focus students' attention to and reasoning about the properties/behaviors of and relationships between operations. Later algebraic work will call upon these properties, behaviors, and relationships. Seeing them first within arithmetic problems makes the transition to algebra less stressful, less of a leap than a step forward.

For instance, understanding "fact families," brings students to understand that if we know $7 \times 3 = 21$ then when asked what is $21 \div 7 =$ ___, we need to think 'what times 7 equals 21?' because we know

that the answer times the divisor yields the dividend; multiplication and division are intimately related. Having this reasoning sequence firmly ensconced in our minds makes questions such as $(x^2 - x - 6) \div (x - 3) =$ ____ easier to solve. We ask ourselves: What would I multiply $(x - 3)$ by to get $(x^2 - x - 6)$?

Russell, Shifter and Bastable also seek to draw students' attention to the regularities about the operational representations. For example, they ask students to consider generalizations about discoveries such as $6 + 4 = 10$ and $4 + 6 = 10$. If that is true, they ask, "Does it work for all '10' combinations?" Does it work for all number combinations? Students of low-level mathematics do not know the answers to these questions right away, as they might their 'facts.' These questions push students to think about the mathematics in new ways, ways that help them uncover the commutative law of addition, for instance, long before they use such a term. Students thus learn to be comfortable "not knowing." They start to view mathematics as multilayered, as something worth analyzing the way they analyze poems or short stories or essays in language classes. In short, argued Russell, they begin to reproduce the behavior of mathematicians in the classroom, just the way they reproduce writers behaviors in the language classroom.

By starting with computations that students know, Russell, Shifter and Bastable found they were able to engage students in discourses about equality and the operation of addition. Russell described how in one class the teacher opened the class by asking individuals to compile a list of pairs of numbers that totaled 32. Students dutifully listed all they could think of – some were organized in their approach; some were not.

However, the teacher was not really interested in anyone listing all the pairs that summed to 32, she wanted to ask them a question to which they did not know the answer.

So, after collecting and recording for the class all the pairs of numbers that summed to 32 that individuals had listed, the teacher chose two sum pairs and asked the question:

> If you don't add each side, how would you know for sure that 16 + 16 = 15 + 17?

Now students had to stop and think. They had to exercise their reasoning ability and communicate their explanations to one another.

How do you know the sums are equal ?

$$16 + 16 = 15 + 17$$

Explain your work below.
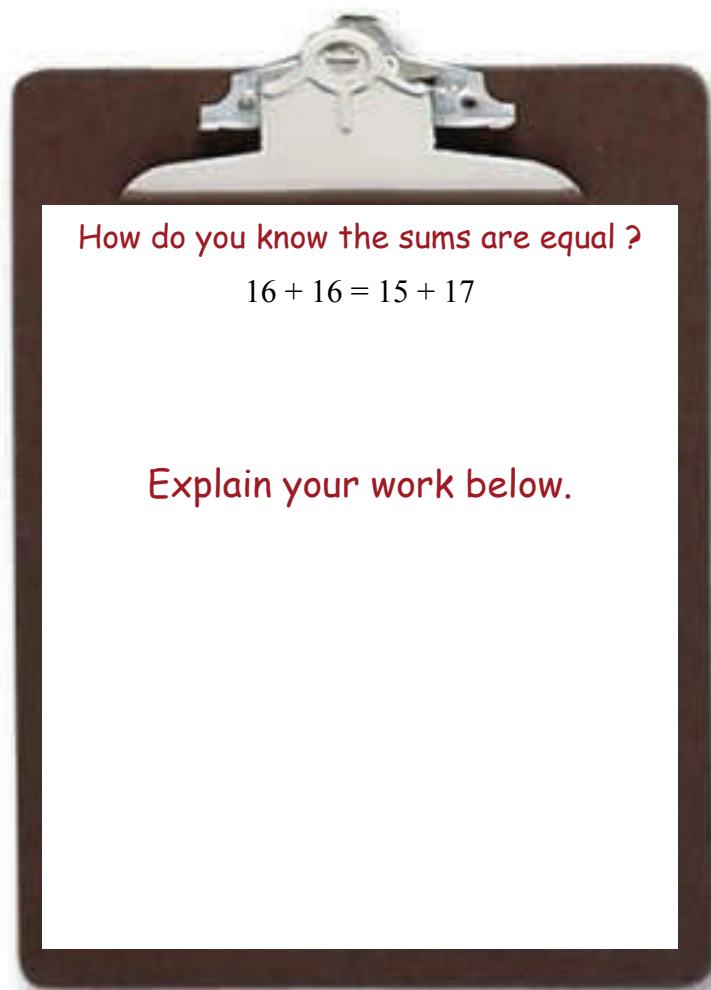
The Habit of Not Knowing: The Key to Reasoning Development

Continued from page 8

A simple computational fluency exercise was about to become a conceptual understanding exploration. As students came to understand that the totals were equal because when you took one away from the first 16 you added one more to the second 16, they began to understand the equation as a balance of values. (This is an essential understanding for work with algebraic equations.)

The students in the '32' class were asked to use cubes to show what they meant when they made statements about taking away and adding (see article on p. 7 about questions to facilitate thinking). As the learners demonstrated and talked through the problem, they came to a generalization that "I can take any number away as long I add it to another; the total remains the same."

> "I can take any number away as long I add it to another; the total remains the same."

One of the regularities of addition surfaced. To keep students thinking like mathematicians, the teacher asked for other examples to be demonstrated.

Students saw that $8 + 8 = 9 + 7$ and that $10 + 10 = 8 + 12$ and so they were guided by examples and demonstrations to own the generalization and to see that it applied in a variety of situations, which is what generalizations do! This was a significant mathematical moment in the class – they had been asked a question to which they did not know the answer, and they had worked their way through the unknowing to come to a place of knowing. Math did make sense! And the idea of equality that they now understood was a much bigger idea than knowing that $14 + 18 = 32$, though the teacher had also reinforced everyone's addition fact knowledge.

16 + 16   =   15 + 17
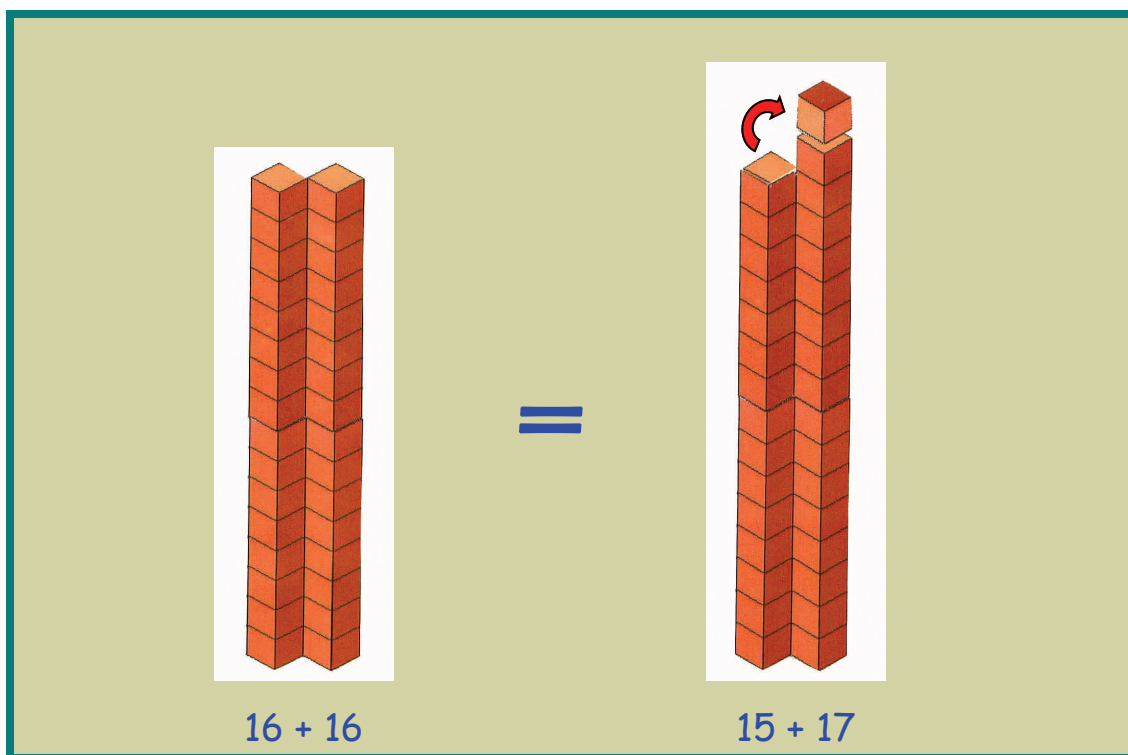
The Habit of Not Knowing: The Key to Reasoning Development

Continued from page 9

In an ABE classroom at the pre-GED or GED level, teachers can try taking this exercise one step further by using algebraic notation to describe the generalization. For instance, the teacher can start by asking how we can write the numbers 15 and 17 using an x, if $x = 16$.

$$16 + 16 = 15 + 17$$

$$x + x = (x - 1) + (x + 1).$$

Students can, with cubes still available, see what is meant when we say that:

$x + x = (x - 1) + (x + 1)$. The verbal generalization started students practicing their algebraic reasoning about the nature of addition. By continuing with the abstraction of the generalization by using variables, the connection to algebraic notation becomes more concrete as well.

Starting with what students know, we create a comfort zone from which exploration into the unknown is more acceptable, according to Russell. She advocated for routine and persistence with these types of questions and dilemmas for students, noting that good students will push harder because they do not like "not knowing" the answers; poor students will eventually master their fear of not knowing because they will realize that it is a temporary state of being, and middle students will wonder, "Why did you ask that?" But because the questions do have answers and the mathematics is significant and meaningful, students do engage to explore the unknown. They become practicing mathematicians comfortable in the "Habit of Not Knowing" along the way.

Editor's Note: Other math questions Russell et.al. asked in order to generate an atmosphere where "The Habit of Unknowing" is the classroom norm will be shared in the spring edition of the *SABES Math Bulletin.*

ACCUPLACER vs. GED:

Different Tests, Different Problems

Adult students in Massachusetts' basic education programs are likely to face two key exams over time: the high school equivalency exam, the GED, and the community college placement test, the ACCUPLACER. Each test includes a unique set of problems, though both extensively use a multiple-choice format.

At an October presentation, Massachusetts GED Chief Examiner, Tom Mecham, shared two problems representative of the differences between the two tests. The problems follow below (answers elsewhere on page).

GED Version of an Area Problem

"Big Papi Ortiz has a beautiful, grassy, rectangular back yard that measures 120 feet by 90 feet. He intends to build a square stone patio in his yard. The patio will measure 60 feet on a side. Once the patio is built, how many square feet of grass will he have in his yard?" (Answer below)

ACCUPLACER Version of an Area Problem

"A rectangle has a length of *2a + 4* and a width of *a – 3*. If the formula for the area of a rectangle is *area = length x width*, what is the area of this rectangle?" (Answer below)

Comparing the Problems

Several differences between the two problems stand out at first glance.

- The GED problem is situated within a context that provides visual clues to its solution.
- The ACCUPLACER problem is stated in mathematical terms, with no context provided.
- The GED problem asks for a solution, a numerical "=___" response.
- The ACCUPLACER problem asks for an algebraic expression response.

More obviously:

- The GED problem uses whole numbers.
- The ACCUPLACER problem uses variables and defines length and width with the same variable.

The exams also exhibit other differences (see p. 12).

A mathematics instructor working to prepare adult basic education students for the two exams, so that they enjoy a variety of options for 'next steps,' needs to remain conscious of the two different types of questions when running her classes. It will help students to see both types of questions, to discuss the differences as they observe them, and to practice with each kind of problem. A simple problem sort, where students separate problems into two piles – GED and ACCUPLACER – can provide an entrée to working on both problem types to ensure learner flexibility.

ACCUPLACER Answer:
$2a^2 - 2a - 12$

GED Answer:
7200 square feet

ACCUPLACER
Continued from page 11

However, when preparing students to be life long mathematics learners, ready for any exam or work challenge, teachers may wish to bear in mind the general pedagogical approach that researchers believe will foster life long learning. This approach involves the nurturing of five processes essential for relating to mathematics (as outlined by the National Research Council in *Adding It Up: Helping Children Learn Mathematics*):

- Conceptual Understanding
- Adaptive Reasoning
- Strategic Competence
- Procedural Fluency
- Productive Disposition

> Teachers want their students to experience success with particular tests, but more importantly, they want their learners to experience being mathematically competent.

To find out more about the five processes outlined above, read NCSALL paper, *The Components of Numeracy* at: http://www.ncsall.net/fileadmin/resources/research/op_numeracy.pdf

ACCUPLACER Test Facts:
(Algebra Section)

- 12 test items covering basic or introductory algebra course topics.
- Three types of questions are asked: solving equations, inequalities and practical problems; operations with algebraic expressions and formulas; and equations with integers and rational numbers. ACCUPLACER algebra questions include linear equations and inequalities, factoring, basic geometry, quadratic equations, exponents, monomials, polynomials, square roots, negative rationals, and absolute values.
- More advanced questions are asked on the College-level Math Section of the ACCUPLACER.
- Use of a calculator is not allowed.
- Test is adaptive and untimed.

GED Mathematics Test Facts:

- Two parts, each with 25 questions; total of 50 questions.
- Part I allows use of the Casio fx260 calculator; Part II does not allow calculator use.
- 80% of the questions use the multiple-choice format; 20% of the test items require test takers to construct their own responses and bubble them in on a standard answer sheet or coordinate grid sheet.
- Test is timed: 45 min./section; total of 90 minutes.
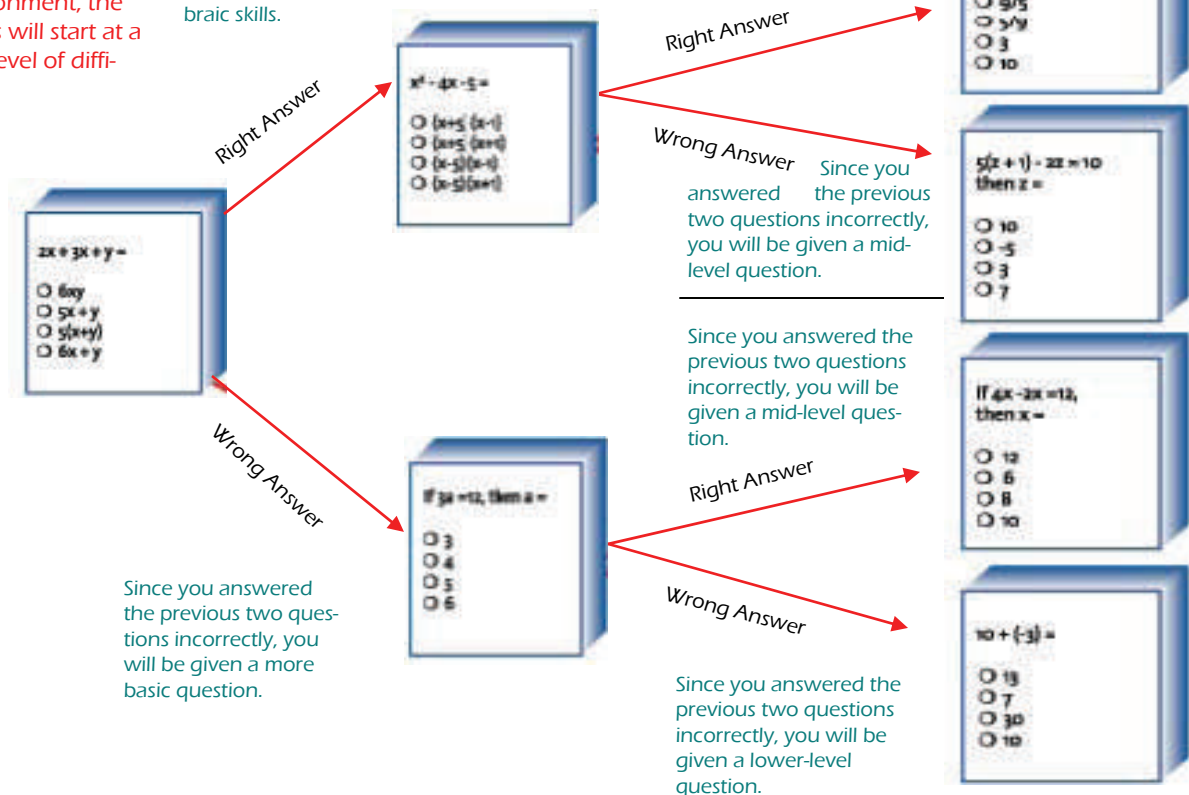- Test is 20-30% algebra-related.

Illustration of Sample Computer-Adaptive Test

The following pool of items is based on elementary algebra test in a computer-adaptive test environment, the test-takers will start at a medium level of difficulty.

Since you answered the previous two questions correctly ,you will be given a question requiring more algebraic skills.

Since you answered the previous two questions correctly , an advanced algebraic question will be presented

Right Answer

Wrong Answer Since you answered the previous two questions incorrectly, you will be given a mid-level question.

Since you answered the previous two questions incorrectly, you will be given a mid-level question.

Right Answer

Wrong Answer

Since you answered the previous two questions incorrectly, you will be given a lower-level question.

Right Answer

Wrong Answer

Since you answered the previous two questions incorrectly, you will be given a more basic question.

For more information about either test and to access sample test questions, simply Google *GED Mathematics Test* or *ACCUPLACER Algebra Test*. For general information about the ACCUPLACER go to : http://www.collegeboard.com/student/testing/ACCUPLACER/

Coming Attractions:

SABES Math Workshops scheduled for 2010

- ❖ Algebra for All
- ❖ Basics of Teaching Math
- ❖ Math for Next Steps: Health Careers
- ❖ Math for Next Steps: Intro to College Level Math

For more information please go to the SABES Calendar at
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